Extra Tutorial 2 (2015 - 2016):

2. Prove that if f is uniformly continuous on a bounded subset A of $\mathbb{R}$, then f is bounded on A. Show that this does not hold for continuous f on A.

5. A function $f: \mathbb{R} \to \mathbb{R}$ is said to be periodic on $\mathbb{R}$ if $\exists\, p > 0$ s.t. $f(x+p) = f(x)$ $\forall x \in \mathbb{R}$. Prove that a continuous, periodic function f on $\mathbb{R}$ is bounded and uniformly continuous on $\mathbb{R}$.

(M1): Does there exist bounded subset A of $\mathbb{R}$, bdd continuous f on A, but f is not uniformly continuous on A?

(M2): Show that if f is uniformly continuous on A, then $\exists$ continuous g on $\bar{A} \stackrel{def}{=} A \cup \{\text{limit points of } A\}$ st $g = f$ on A. Such g is unique and g is uniformly continuous on $\bar{A}$.

(M3): Let f be monotone function on $\mathbb{R}$. Show that $\{$discontinuous points of $f\}$ is countable. Assuming f is increasing

(i) Observe that f is discontinuous at a iff $\lim_{x\to a^-} f(x) \neq \lim_{x \to a^+} f(x)$ where both limits must exist.

(ii) Observe that if f is discontinuous at a and b with $a \neq b$, then

$$\left(\lim_{x\to a^-} f(x), \lim_{x\to a^+} f(x)\right) \cap \left(\lim_{x\to b^-} f(x), \lim_{x\to b^+} f(x)\right) = \phi$$

Hence, we can define 1-1 map from $\{$discontinuous points of $f\}$ to $\mathbb{Q}$.

(M4) Let $f: (a,b) \to \mathbb{R}$. We say that f satisfies intermediate value Property if whenever $\exists\, k$ st. $f(x_1) < k < f(x_2)$ for some x_1, x_2, then $\exists\, c$ bet' x_1, x_2 s.t. $f(c) = k$.

Show that if f is monotone and satisfies Intermediate value property, then f is continuous.

Exam (2015 - 2016)

4. Let $f: \mathbb{R} \to \mathbb{R}$ be continuous st. $\lim_{x\to -\infty} f(x) = l$ and $\lim_{x \to +\infty} f(x) = L$ exist in $\mathbb{R}$.

(b) Show that f is uniformly continuous on $\mathbb{R}$.

4(c) Can the condition $\lim_{x \to +\infty} f(x) = L$ be dropped for (b)? Provide your reasoning.

Now, by Q4, show by example that uniformly continuous odd fcn f need not attain global maximum nor global minimum.

Exam (2014-2015).

6. Let $n \in \mathbb{N} \setminus \{1\}$ and let the continuous monotone function $f_n : [\frac{1}{2}, 1] \longrightarrow \mathbb{R}$.

be defined by $f_n(x) = x^n + x \quad \forall \ x \in [\frac{1}{2}, 1]$.

(a) Show that for each n, there exists one and only one root z_n of the equation $f_n(x) = 1$ s.t. $z_n^n + z_n = 1$

(b) Show further that $\lim_n z_n$ exists in $\mathbb{R}$. Can you determine the value of the limit? Why?

Solution :

Extra Tutorial 2 (2015-2016)

Q2 Method 1 : Suppose not. f is unbdd on A. $\exists$ seq $\{x_n\}_{n=1}^{\infty} \subset A$ st. $|f(x_n)| \geq n$. $\forall n \in \mathbb{N}$. Since A is bdd, by Bolzano Weierstrass thm, $\exists$ subseq $\{x_{n_k}\}_{k=1}^{\infty}$ st. x_{n_k} converges as $k \to \infty$, say with limit z. We don't know whether $z \in A$. ($z \notin A$ in general)

Nonetheless, it is a Cauchy sequence in A: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ st. $|x_{n_i} - x_{n_j}| < \varepsilon$ whenever $i,j \geq K$.

By uniform continuity of f on A. $\exists \varepsilon > 0$ st. $|f(x) - f(y)| < 1$ $\forall x,y \in A$ with $|x-y| < \varepsilon$ — ①

For such $\varepsilon > 0$, $\exists K \in \mathbb{N}$ st. $|x_{n_i} - x_{n_j}| < \varepsilon$ $\forall$ $i,j \geq K$ — ②.

Fix x_{n_K}, $\exists P \in \mathbb{N}$ st. $|f(x_{n_K})| < P - 1$, then $P \geq n_K \geq K$ and by ②.

we have $|x_{n_K} - x_{n_P}| < \varepsilon$. But now, $|f(x_{n_K}) - f(x_{n_P})| \geq |f(x_{n_P})| - |f(x_{n_K})|$

$\geq P - |f(x_{n_K})| > 1$

Contradicts ① because both $x_{n_K}, x_{n_P} \in A$.

Method 2 : By uniform continuity of f on A, $\exists \varepsilon > 0$ st. $|f(x) - f(y)| < 1$ whenever $x, y \in A$ with $|x-y| < \varepsilon$. Now it suffices to show that there is a finite subcover of $\{(x-\varepsilon, x+\varepsilon) : x \in A\}$ for A. This is done if A is closed. Now consider $\bar{A} := A \cup \{\text{limit points of } A\}$, since $\bar{A}$ is compact (closed and bdd) there is a finite subcover of $\{(x-\varepsilon, x+\varepsilon) : x \in A\}$ for $\bar{A}$ if $\{(x-\varepsilon, x+\varepsilon) : x \in A\}$ covers $\bar{A}$. Finite subcover for $\bar{A}$ is finite subcover for A. Hence the problem reduces to show that $\{(x-\varepsilon, x+\varepsilon) : x \in A\}$ covers $\bar{A}$.

Let $z \in \bar{A}$, $\exists y \in A$ st. $|y-z| < \varepsilon$. Now, $z \in (y-\varepsilon, y+\varepsilon)$

$\therefore$ $\bar{A} \subset \bigcup_{y \in A} (y-\varepsilon, y+\varepsilon)$ which is equivalent to say $\{(x-\varepsilon, x+\varepsilon) : x \in A\}$ covers $\bar{A}$.

Q3. For boldness of f, it suffices to show that $f(\mathbb{R}) = f([0,p])$.

For the inclusion $\subset$, let $x \in \mathbb{R}$, by Well ordering property of $\mathbb{Z}$, there is $n = \max\{k \in \mathbb{Z} : x > kp\}$, hence $np < x \leq (n+1)p$ and $x - np \in (0, p]$

$\therefore$ $f(x) = f(x-np) \in f([0,p])$.

For uniform continuity of f on $\mathbb{R}$:

Since f is continuous on $[-2p, 2p]$, f is uniform continuous on $[-2p, 2p]$:

Let $\varepsilon > 0$, $\exists \delta > 0$ st. $|f(x) - f(y)| < \varepsilon$ $\forall x,y \in [-2p, 2p]$, $|x-y| < \delta$, we can take δ to be $\delta < p$. — ①

For such $\delta > 0$, let $z \in \mathbb{R}$; $\exists\, n \in \mathbb{Z}$ s.t. $z - np \in [0, p]$. if $|z - w| < \delta$, then

$(w - np) \in [-2p, 2p]$, by ①, we have $|f(z - np) - f(w - np)| < \varepsilon$.

But now $|f(z) - f(w)| = |f(z - np) - f(w - np)|$ $\therefore$ f is uniformly continuous on $\mathbb{R}$.

<u>(M2)</u> : let $x \in \bar{A} \setminus A$, let $\{x_n\}_{n=1}^{\infty} \subset A$ st $x_n \to x$ as $n \to \infty$

then $\{x_n\}_{n=1}^{\infty}$ is Cauchy, by uniform continuity of f on A, $\{f(x_n)\}_{n=1}^{\infty}$ is Cauchy. (This needs verification. left to you)

Hence $\{f(x_n)\}_{n=1}^{\infty}$ converges. By sequential criterion, $\lim_{y \to x} f(y)$ exists

Now define $g(c) = \begin{cases} f(c) & \text{if } c \in A \\ \lim_{x \to c} f(x) & \text{if } c \in \bar{A} \setminus A \end{cases}$. It should be noted that if h is

a continuous function on $\bar{A}$ st. $h = f$ on A. then $h(c) = \lim_{\substack{x \to c \\ x \in \bar{A}}} h(x) = \lim_{\substack{x \to c \\ x \in A}} h(x) = \lim_{x \to c} f(x)$

for $c \in \bar{A} \setminus A$. $\therefore$ You have no choice to define g.

We show that g is uniformly continuous on $\bar{A}$, hence continuous on $\bar{A}$.

Since domain of $g \neq A$, f continuous at $x \in A$ and $g = f$ on A $\not\Rightarrow$ g

continuous at $x \in A$.

let $\varepsilon > 0$, by uniform continuity of f on A, $\exists\, \delta > 0$ st. $|f(x) - f(y)| < \varepsilon$

whenever $x, y \in A$ with $|x - y| < \delta$. ——— (☆)

Now let $z \in \bar{A}$, $w \in \bar{A}$ s.t. $|w - z| < \delta/3$,

$\exists\, z_1 \in A$ st. $|z_1 - z| < \delta/3$ and $|f(z_1) - g(z)| < \varepsilon$

$\exists\, w_1 \in A$ st. $|w_1 - w| < \delta/3$ and $|f(w_1) - g(w)| < \varepsilon$

Hence $|g(w) - g(z)| \leq |g(w) - f(w_1)| + |f(w_1) - f(z_1)| + |f(z_1) - g(z)|$

$< \varepsilon + |f(w_1) - f(z_1)| + \varepsilon$

Also note both $w_1, z_1 \in A$ and $|z_1 - w_1| \leq |z_1 - z| + |z - w| + |w - w_1|$

$< \delta/3 + \delta/3 + \delta/3 = \delta$

$\therefore$ $|f(w_1) - f(z_1)| < \varepsilon$ by (☆)

for short, let $\varepsilon > 0$ let $z \in \bar{A}$, $w \in \bar{A}$ st. $|w - z| < \delta/3$, we have $|g(w) - g(z)| < 3\varepsilon$

$\therefore$ g is uniformly continuous on $\bar{A}$.

(M4). Let $x \in (a,b)$ let $\varepsilon > 0$, assuming f is increasing

For $x \in (a,b)$, $\exists\ x_1, x_2 \in (a,b)$ s.t. $x_1 < x < x_2$

If $f(x)-\varepsilon < f(x_1)$, then we are done. If $f(x)-\varepsilon \geq f(x_1)$, then $f(x) > f(x_1)$

By intermediate value Property, $\exists\ x_1' \in (x_1, x)$ s.t. $f(x_1') = \max\left\{ \frac{f(x)+f(x_1)}{2}, f(x) - \frac{\varepsilon}{2} \right\}$

So, $\exists\ x_1' < x$ s.t. $f(x) - f(x_1') < \varepsilon$

Similarly, $\exists\ x_2' > x$ s.t. $f(x_2') - f(x) < \varepsilon$

Let $\delta := \min\{(x - x_1'), (x_2' - x)\}$ (>0), by monotonicity of f,

$\forall\ y \in (x-\delta, x+\delta)$, $\wedge$ $-\varepsilon < f(x) - f(x_2') \leq f(x) - f(y) \leq f(x) - f(x_1') \leq \frac{\varepsilon}{2} < \varepsilon$

$y \in (x_1', x_2')$, hence

$\therefore$ f is continuous at x.

For f decreasing, consider $(-f)$

Exam (2015 - 2016)

4(b) Let $\varepsilon > 0$, Since $\lim_{x \to -\infty} f(x) = l$, $\lim_{x \to +\infty} f(x) = 2$ exist in $\mathbb{R}$.

$\exists\ M > 1000$ s.t. $|f(x) - l| < \frac{\varepsilon}{2}$ $\forall\ x < -M$

$|f(x) - 2| < \varepsilon/2$ $\forall\ x > M$

$\therefore$ $|f(x) - f(y)| \leq |f(x) - l| + |l - f(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. $\forall\ x, y < -M$

$|f(x) - f(y)| \leq |f(x) - 2| + |2 - f(y)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ $\forall\ x, y > M$

Since f is continuous on $[-M-2, M+2]$, f is uniformly continuous

on $[-M-2, M+2]$. $\exists\ \delta > 0$ s.t. $|f(x) - f(y)| < \varepsilon$ $\forall\ x, y \in [-M-2, M+2]$ with $|x-y| < \delta$

We can take δ to be < 1

For such $\delta > 0$, for $z \in \mathbb{R}$, if $|w - z| < \delta$, we have three cases

1. $z < -M-1$, in this case, both $z, w < -M$
2. $-M-1 \leq z \leq M+1$, in this case, both $z, w \in [-M-2, M+2]$, $|z-w| < \delta$
3. $M+1 < z$, in this case, both $z, w > M$

In any cases, $|f(z) - f(w)| < \varepsilon$ $\therefore$ f is uniformly continuous on $\mathbb{R}$.

Q6 (a). For $n \in \mathbb{N} \setminus \{1\}$,

Since $f_n(\frac{1}{2}) = (\frac{1}{2})^n + \frac{1}{2} \le (\frac{1}{2})^2 + \frac{1}{2} = \frac{3}{4} < 1$ and $f_n(1) := 1^n + 1 = 2 > 1$ and

f_n is continuous on $[\frac{1}{2}, 1]$, by intermediate value theorem, $\exists\, z_n \in (\frac{1}{2}, 1)$

s.t. $f_n(z_n) = 1$,

Since f_n is strictly increasing on $[\frac{1}{2}, 1]$, such root is unique.

(b) Now $z_n^n + z_n = 1$. Check if $\{z_n\}_{n=1}^{\infty}$ is monotone.

$z_{n+1}^{n+1} + z_{n+1} = 1 = z_n^n + z_n \Rightarrow z_{n+1} - z_n = z_n^n - z_{n+1}^{n+1} > z_n^n - z_{n+1}^n$

because $z_{n+1} \in (\frac{1}{2}, 1)$. Suppose $z_{n+1} \le z_n$, then $z_{n+1} - z_n > z_n^n - z_{n+1}^n \ge 0$

$\Rightarrow z_{n+1} > z_n$. Contradiction. $\therefore z_{n+1} > z_n$ and $\{z_n\}_{n=1}^{\infty}$ is increasing

By MCT, $\lim_{n\to\infty} z_n$ exists (Here we used $z_n \in [\frac{1}{2}, 1]$ $\forall n \in \mathbb{N}$).

Suppose $z := \lim_{n\to\infty} z_n < 1$, $0 \le z_n \le z$ $\forall n \in \mathbb{N}$

$\Rightarrow 0 \le z_n^n \le z^n$, where $\lim_{n\to\infty} z^n = 0$

$\therefore \lim_{n\to\infty} z_n^n = 0$ by Sandwich thm.

Since $z_n^n + z_n = 1$. $\forall n \in \mathbb{N}$, $1 = \lim_{n\to\infty} (z_n^n + z_n) = 0 + z$

Contradicts to $z < 1$ $\therefore z \ge 1$ and note $z \in [\frac{1}{2}, 1]$.

we have $z = 1$.